Selected Solutions Manual

for

Introduction to Partial Differential Equations

by

Peter J. Olver

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Selected Solutions to
Chapter 1: What Are Partial Differential Equations?

1.1. (a) Ordinary differential equation, equilibrium, order = 1;

(c) partial differential equation, dynamic, order = 2;

(e) partial differential equation, equilibrium, order = 2;

(g) partial differential equation, equilibrium, order = 2;

(i) partial differential equation, dynamic, order = 3;

(k) partial differential equation, dynamic, order = 4.

1.2. (a) (i) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, (ii) $u_{xx}+u_{yy}=0$; (c) (i) $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$, (ii) $u_t=u_{xx}+u_{yy}$;

1.4. (b) independent variables: x, y; dependent variables: u, v; order = 2;

(d) independent variables: t, x, y; dependent variables: u, v, p; order = 1.

1.5.

(a) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=e^x\cos y-e^x\cos y=0$; defined and C^∞ on all of $\mathbb{R}^2$.

(c) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=6x-6x=0$; defined and C^∞ on all of $\mathbb{R}^2$.

(d) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{2y^2-2x^2}{(x^2+y^2)^2}+\dfrac{2x^2-2y^2}{(x^2+y^2)^2}=0$; defined and C^∞ on $\mathbb{R}^2\setminus\{0\}$.

1.7. $u=\log\big[c(x-a)^2+c(y-b)^2\big]$, for a, b, c arbitrary constants.

1.8. (a) $c_0+c_1x+c_2y+c_3z+c_4(x^2-y^2)+c_5(x^2-z^2)+c_6xy+c_7xz+c_8yz$, where $c_0,\ldots,c_8$ are arbitrary constants.

1.10. (a) $\dfrac{\partial^2 u}{\partial t^2}-4\dfrac{\partial^2 u}{\partial x^2}=8-8=0$; (c) $\dfrac{\partial^2 u}{\partial t^2}-4\dfrac{\partial^2 u}{\partial x^2}=-4\sin 2t\cos x+4\sin 2t\cos x=0$.

1.11. (a) $c_0+c_1t+c_2x+c_3(t^2+x^2)+c_4tx$, where $c_0,\ldots,c_4$ are arbitrary constants.

1.13. $u=a+\dfrac{b}{r}=a+\dfrac{b}{\sqrt{x^2+y^2+z^2}}$, where a, b are arbitrary constants.

1.15. Example: (b) $u_x^2+u_y^2+u^2=0$ — the only real solution is $u\equiv 0$.

1.16. When $(x,y) \neq (0,0)$, a direct computation shows that

$$\frac{\partial u}{\partial x} = \frac{y\,(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}, \qquad \frac{\partial u}{\partial y} = \frac{x\,(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2},$$

while, from the definition of partial derivative,

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h \to 0} \frac{u(h,0)-u(0,0)}{h} = 0, \qquad \frac{\partial u}{\partial y}(0,0) = \lim_{k \to 0} \frac{u(0,k)-u(0,0)}{k} = 0.$$

Thus,

$$\frac{\partial^2 u}{\partial x\,\partial y}(0,0) = \lim_{h \to 0} \frac{u_y(h,0)-u_y(0,0)}{h} = 1, \qquad \frac{\partial^2 u}{\partial y\,\partial x}(0,0) = \lim_{k \to 0} \frac{u_x(0,k)-u_x(0,0)}{k} = -1.$$

This does not contradict the equality of mixed partials because the theorem requires continuity, while

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial x\,\partial y} = \frac{x^6 + 9x^4y^2 + 9x^2y^4 - y^4}{(x^2+y^2)^3}, \qquad (x,y) \neq (0,0),$$

is not continuous at $(x,y) = (0,0)$. Indeed,

$$\lim_{h \to 0} \frac{\partial^2 u}{\partial x\,\partial y}(h,0) = 1 \neq -1 = \lim_{k \to 0} \frac{\partial^2 u}{\partial x\,\partial y}(0,k).$$

1.17. (a) homogeneous linear; (*d*) nonlinear; (*f*) inhomogeneous linear.

1.19. (a) (*i*) $\dfrac{\partial^2 u}{\partial t^2} = -4\cos(x-2t) = 4\,\dfrac{\partial^2 u}{\partial x^2}$.

1.20. (a) $\cos(x-2t) + \frac{1}{4}\cos x - 5\sin(x-2t) - \frac{5}{4}\sin x$.

1.21. (a) $\partial_x[c f + d g] = \dfrac{\partial}{\partial x}[c f(x) + d g(x)] = c\,\dfrac{\partial f}{\partial x} + d\,\dfrac{\partial g}{\partial x} = c\partial_x[f] + d\partial_x[g]$. The same proof works for ∂_y. (*b*) Linearity requires $d = 0$, while a, b, c can be arbitrary functions of x, y.

1.23. Using standard vector calculus identities:

(*b*) $\nabla \times (\mathbf{f} + \mathbf{g}) = \nabla \times \mathbf{f} + \nabla \times \mathbf{g}$, $\nabla \times (c\mathbf{f}) = c\nabla \times \mathbf{f}$.

1.24. (a)
$$\begin{aligned}(L-M)[u+v] &= L[u+v] - M[u+v] = L[u] + M[u] - L[v] - M[v] \\ &= (L-M)[u] + (L-M)[v],\end{aligned}$$
$$(L-M)[cu] = L[cu] - M[cu] = cL[u] - cM[u] = c(L-M)[u];$$

(*c*)
$$(f\,L)[u+v] = f\,L[u+v] = f\,L[u] + f\,L[v] = (f\,L)[u] + (f\,L)[v],$$
$$(f\,L)[cu] = f\,L[cu] = f\,cL[u] = c(f\,L)[u].$$

1.27. (*b*) $u(x) = \frac{1}{6}e^x \sin x + c_1 e^{2x/5}\cos \frac{4}{5}x + c_2 e^{2x/5}\sin \frac{4}{5}x$.

1.28. (*b*) $u(x) = -\frac{1}{9}x - \frac{1}{10}\sin x + c_1 e^{3x} + c_2 e^{-3x}$,

(*d*) $u(x) = \frac{1}{6}x e^x - \frac{1}{18}e^x + \frac{1}{4}e^{-x} + c_1 e^x + c_2 e^{-2x}$.

Selected Solutions to
Chapter 2: Linear and Nonlinear Waves

2.1.1. $u(t,x) = t\,x + f(x)$, where f is an arbitrary C^1 function.

2.1.3. (a) $u(t,x) = f(t)$; (c) $u(t,x) = t\,x - \frac{1}{2}t^2 + f(x)$; (e) $u(t,x) = e^{-t\,x} f(t)$.

2.1.5. $u(t,x,y) = f(x,y)$ where f is an arbitrary C^1 function of two variables. This is valid provided each slice $D_{a,b} = D \cap \{\,(t,a,b) \mid t \in \mathbb{R}\,\}$, for fixed $(a,b) \in \mathbb{R}^2$, is either empty or a connected interval.

♡ 2.1.8. (a) The partial differential equation is really an autonomous first-order ordinary differential equation in t, with x as a parameter. Solving this ordinary differential equation by standard methods, [**20**, **23**], the solution to the initial value problem is $u(t,x) = \dfrac{f(x)}{t\,f(x)+1}$. Thus, if $f(x) > 0$, then the denominator does not vanish for $t \geq 0$, and, moreover, goes to ∞ as $t \to \infty$. Therefore, $u(t,x) \to 0$ as $t \to \infty$.

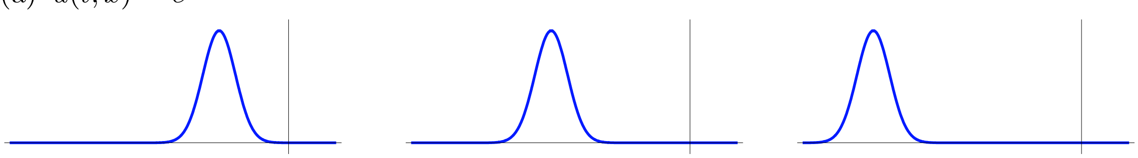
(b) If $f(x) < 0$, then the denominator in the preceding solution formula vanishes when $t = \tau = -1/f(x)$. Moreover, for $t < \tau$, the numerator is negative, while the denominator is positive, and so $\lim\limits_{t \to \tau^-} u(t,x) = -\infty$.

(c) The solution is defined for $0 < t < t_\star$, where $t_\star = -1/\min f(x)$. In particular, if $\min f(x) = -\infty$, then the solution is not defined for all $x \in \mathbb{R}$ for *any* $t > 0$.

♢ 2.1.9. It suffices to show that, given two points (t_1,x), $(t_2,x) \in D$, then $u(t_1,x) = u(t_2,x)$. By the assumption, $(t,x) \in D$ for $t_1 \leq t \leq t_2$, and so $u(t,x)$ is defined and continuously differentiable at such points. Thus, by the Fundamental Theorem of Calculus,

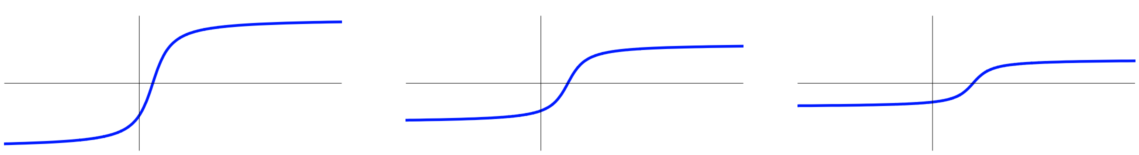
$$u(t_2,x) - u(t_1,x) = \int_{t_1}^{t_2} \frac{\partial u}{\partial t}\,(s,x)\,ds = 0. \qquad Q.E.D.$$

2.2.2. (a) $u(t,x) = e^{-(x+3\,t)^2}$



$t = 1$ $\qquad$ $t = 2$ $\qquad$ $t = 3$

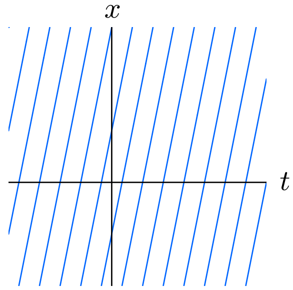
(c) $u(t,x) = e^{-t/2}\tan^{-1}(x-t)$



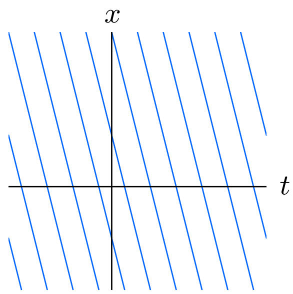
$t = 1$ $\qquad$ $t = 2$ $\qquad$ $t = 3$

2.2.3.

(b) Characteristic lines: $x = 5t + c$; general solution: $u(t,x) = f(x - 5t)$;



(d) Characteristic lines: $x = -4t + c$; general solution: $u(t,x) = e^{-t} f(x + 4t)$;



2.2.4. $u(t,x) = t + e^{-(x-2t)^2}$.

♢ 2.2.6. By the chain rule

$$\frac{\partial v}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t - t_0, x), \qquad \frac{\partial v}{\partial x}(t,x) = \frac{\partial u}{\partial x}(t - t_0, x),$$

and hence

$$\frac{\partial v}{\partial t}(t,x) + c\,\frac{\partial v}{\partial x}(t,x) = \frac{\partial u}{\partial t}(t - t_0, x) + c\,\frac{\partial u}{\partial x}(t - t_0, x) = 0.$$

Moreover, $v(t_0, x) = u(0,x) = f(x)$. *Q.E.D.*

2.2.9. (a) $|\,u(t,x)\,| = |\,f(x - ct)\,|\,e^{-at} \le M\,e^{-at} \to 0$ as $t \to \infty$ since $a > 0$.

(b) For example, if $c \ge a$, then the solution $u(t,x) = e^{(c-a)t-x} \not\to 0$ as $t \to \infty$.

♡ 2.2.11. (a) $u(t,x) = \dfrac{1}{t + h(x-t)}$, where $h(\xi)$ is an arbitrary C^1 function.

(b) The solution to the initial value problem is

$$u(t,x) = \frac{f(x-t)}{1 + t\,f(x-t)}.$$

If $f(x) \ge 0$, then the denominator does not vanish for $t \ge 0$, and hence the solution exists for $t > 0$. Moreover, for fixed $t > 0$, the function $g(y) = \dfrac{y}{1 + ty}$ is strictly increasing for $y \ge 0$. Therefore, $0 \le u(t,x) \le \dfrac{M}{1 + M\,t} \longrightarrow 0$ as $t \to \infty$.

(c) Using the preceding solution formula, if $f(x) < 0$, then, at the point $x_\star = x - 1/f(x)$, the solution $u(t, x_\star) \to -\infty$ as $t \to \tau = -1/f(x)$.

(d) If $m = \min f(x) < 0$, then, by part (c), the minimal blow-up time is $\tau_\star = -1/m$.

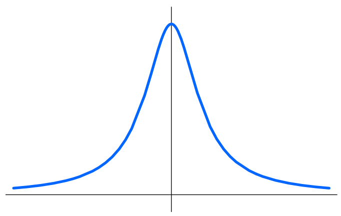
2.2.14. (a) $u(t,x) = \begin{cases} f(x - c\,t), & x \geq c\,t, \\ g(t - x/c), & x \leq c\,t, \end{cases}$ defines a classical C^1 solution provided the compatibility conditions $g(0) = f(0)$, $g'(0) = -c\,f(0)$, hold.

(b) The initial condition affects the solution for $x \geq c\,t$, whereas the boundary condition affects the solution for $x \leq c\,t$. Apart from the compatibility condition along the characteristic line $x = ct$, they do not affect each other.

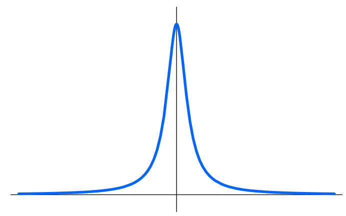
2.2.17. (a) $u(t,x) = \dfrac{1}{(x\,e^t)^2 + 1} = \dfrac{e^{-2t}}{x^2 + e^{-2t}}$.

(b)

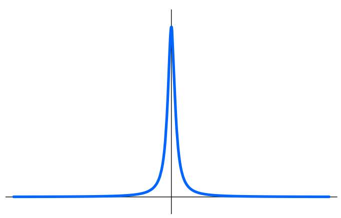
$t = 0$:



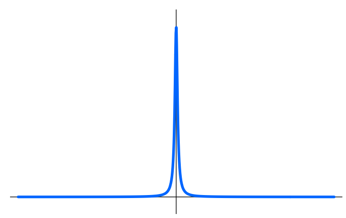
$t = 1$:



$t = 2$:



$t = 3$:



(c) The limit is discontinuous: $\displaystyle \lim_{t \to \infty} u(t,x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$

2.2.18. (a) $\displaystyle \lim_{t \to \infty} u(t,x) = \begin{cases} 0, & x < -1, \\ f(-1), & x = -1, \\ f(1), & x > -1. \end{cases}$

2.2.20. (a) The characteristic curves are given by $x = \tan(t + k)$ for $k \in \mathbb{R}$.